Chapter 3

INTRODUCTION TO CALCULUS

As a child, you learned how to communicate through words. You saw the cookie in the jar, pointed, and said, "Cookie!" Gradually, you learned to compose these words into sentences to more clearly communicate your meaning. As you got older, you studied the rules of grammar, and you learned that there was a correct way to form these sentences, with rules to follow.

Calculus developed in a very similar way: Sir Isaac Newton and Gottfried Wilhelm von Leibniz independently organized an assortment of ideas and methods that were circulating among the mathematicians of their time. As a tool in the service of science, calculus served its purpose very well, but it took over two centuries for mathematicians to identify and agree on its underlying principles—its grammar. In this chapter, you will see some of the ideas that were brought together to form the underlying principles of calculus.

CHAPTER EXPECTATIONS In this chapter, you will

- determine the equation of the tangent to a graph, **Section 3.1**
- understand the slope of the tangent to a curve, **Section 3.1**
- pose problems and formulate hypotheses regarding rates of change, **Section 3.2, Career Link**
- calculate and interpret average rates of change, **Section 3.2**
- estimate and interpret instantaneous rates of change, **Section 3.2**
- explain the difference between average and instantaneous rates of change, **Section 3.2**
- understand the slope of a secant on a curve and the slope of the tangent to a curve, **Section 3.1, 3.2, 3.4**
- make inferences from models of applications and compare the inferences with the original hypotheses regarding the rates of change, **Section 3.2**
- understand the instantaneous rate of change of a function, **Section 3.3, 3.4**
- determine properties of the graphs of polynomial functions, **Section 3.5**
- identify discontinuous functions, **Section 3.5**

Review of Prerequisite Skills

Before beginning this Introduction to Calculus chapter, you may wish to review the following concepts from previous courses and chapters:

- Determining the slope of a line
- Determining the equation of a line
- Using function notation for substituting into and evaluating functions
- Simplifying a radical expression with a monomial or a binomial radical in the denominator
- Factoring expressions
- Finding the domain of functions

Exercise

1. Determine the slope of the line passing through each of the following pairs of points:

   a. $(2, 5)$ and $(6, -7)$
   b. $(3, -4)$ and $(-1, 4)$
   c. $(-6, -1)$ and $(-5, 11)$
   d. $(0, 10)$ and $(10, 0)$
   e. $(-3, 6)$ and $(3, 2)$
   f. $(-3, 6)$ and $(6, 0)$
   g. though the origin and $(1, 4)$
   h. though the origin and $(-1, 4)$
   i. $(0, 1)$ and $(-6, 6)$
   j. $(-2, 4)$ and $(-6, 8)$
   k. $(-2.1, 4.41)$ and $(-2, 4)$
   l. $\left(\frac{3}{4}, \frac{1}{4}\right)$ and $\left(\frac{7}{4}, -\frac{3}{4}\right)$

2. Find the equation of a line determined by the given information.

   a. slope 4, y-intercept -2
   b. slope -2, y-intercept 5
   c. slope 0, y-intercept -5
   d. slope $\frac{2}{3}$, y-intercept 4
   e. through $(-1, 6)$ and $(4, 12)$
   f. through $(-2, 4)$ and $(-6, 8)$
   g. through $(0, 2)$ and $(-1, -4)$
   h. through the origin and $(-1, -4)$
   i. slope 7, through $(4, 1)$
   j. slope -3, through $(1, 3)$
   k. vertical, through $(-3, 5)$
   l. horizontal, through $(-3, 5)$

3. The domain of a function f is the set of all numbers x, and its values are given by $f(x) = \frac{x}{x^2 + 4}$. Find each of the following values:

   a. $f(-10)$
   b. $f(-3)$
   c. $f(0)$
   d. $f(10)$

4. A function f is defined for all x, and its values are given by

$$f(x) = \begin{cases} \sqrt{3-x}, \text{ if } x < 0. \\ \sqrt{3+x}, \text{ if } x \geq 0. \end{cases}$$

Compute each of the following:

a. $f(-33)$ b. $f(0)$ c. $f(78)$

5. A function s is defined for $t > -3$ by $s(t) = \begin{cases} \frac{1}{t}, \text{ if } -3 < t < 0. \\ 5, \text{ if } t = 0. \\ t^3, \text{ if } t > 0. \end{cases}$

Find each of the following:

a. $s(-2)$ b. $s(-1)$ c. $s(0)$ d. $s(1)$ e. $s(100)$

6. Rationalize each denominator.

a. $\frac{5}{\sqrt{2}}$ b. $\frac{6+\sqrt{2}}{\sqrt{3}}$ c. $\frac{2\sqrt{3}+4}{\sqrt{3}}$

d. $\frac{1}{3+\sqrt{3}}$ e. $\frac{5}{\sqrt{7}-4}$ f. $\frac{2\sqrt{3}}{\sqrt{3}-2}$

g. $\frac{5\sqrt{3}}{2\sqrt{3}+4}$ h. $\frac{3\sqrt{2}}{2\sqrt{3}-5}$ i. $\frac{2\sqrt{5}}{2\sqrt{5}-1}$

7. Rationalize each numerator.

a. $\frac{\sqrt{2}}{5}$ b. $\frac{\sqrt{3}}{6+\sqrt{2}}$ c. $\frac{\sqrt{7}-4}{5}$

d. $\frac{2\sqrt{3}-5}{3\sqrt{2}}$ e. $\frac{\sqrt{3}-\sqrt{7}}{4}$ f. $\frac{2\sqrt{3}+\sqrt{7}}{5}$

8. Factor each of the following:

a. $x^2 - 4$ b. $x^3 - x$ c. $x^2 + x - 6$

d. $2x^2 - 7x + 6$ e. $x^3 + 2x^2 + x$ f. $x^3 + 8$

g. $27x^3 - 64$ h. $x^3 - 2x^2 + 3x - 6$ i. $2x^3 - x^2 - 7x + 6$

9. What is the domain of each of the following?

a. $3x + 2y - 7 = 0$ b. $y = x^2$ c. $y = \sqrt{x+5}$

d. $y = x^3$ e. $y = \frac{3}{x-1}$ f. $y = 4x + 5$

g. $y = -\sqrt{x-9} + 7$ h. $y = \frac{x^2+4}{x}$ i. $y = \frac{4}{5-x}$

j. $y = \frac{7}{x^2 - 3x - 4}$ k. $y = \frac{6x}{2x^2 - 5x - 3}$

l. $y = \frac{(x-3)(x+4)}{(x+2)(x-1)(x+5)}$

Career Link investigate

CHAPTER 3: ASSESSING ATHLETIC PERFORMANCE

Differential calculus is fundamentally about the idea of *instantaneous rate of change*. A rate of change familiar to us is "heart rate." Elite athletes are keenly interested in the analysis of heart rates. Obviously, sporting performance is enhanced when an athlete is able to increase his or her heart rate at a slower pace (i.e., get tired less quickly). Heart rate is the *rate of change* of the number of heartbeats with respect to time. A heart rate is given for an *instant* in time. In calculus terminology, heart rate at an instant in time is known as the instantaneous rate of change of the number of heartbeats with respect to time. When a nurse or doctor counts our heartbeats then divides by the time elapsed, they are *not* determining the instantaneous rate of change but instead are calculating the average heart rate over a period of time (usually ten seconds). In this chapter, the idea of the **derivative** will be developed, progressing from the average rate of change being calculated over a smaller and smaller interval until a limiting value is reached at the instantaneous rate of change.

Time (in seconds)	Number of Heartbeats
10, 0.17	9
20, 0.33	19
30, 0.50	31
40, 0.67	44
50, 0.83	59
60, 1.00	75

Case Study—Assessing Elite Athlete Performance

The table below shows the number of heartbeats of an athlete who is undergoing a cardio-vascular fitness test. Complete the discussion questions to determine if this athlete is under his or her maximum desired heart rate of 65 beats per minute at precisely 30 seconds.

DISCUSSION QUESTIONS

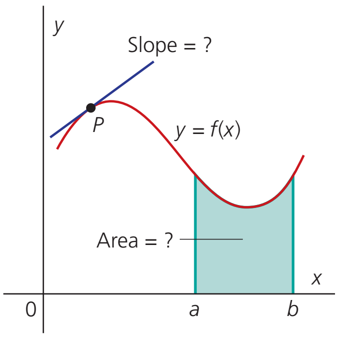
1. Graph the number of heartbeats versus time (in minutes) on graph paper, joining the points to make a smooth curve. Draw a second relationship on the same set of axes showing the resting heart rate of 50 beats per minute. Use the slopes of the two relationships graphed to explain why the test results indicate that the person must be exercising.
2. Discuss how the average heart rate between two points in time could be calculated on this graph. Explain your reasoning.
3. Calculate the athlete's average heart rate at $t = 30$ s over the intervals of [0 s, 60 s], [10 s, 50 s] and [20 s, 40 s]. Show the progression of these average speed calculations on the graph as a series of secants.
4. Use the progression of your average heart rate secants to make a graphical prediction of the instantaneous heart rate at $t = 30$ s. Is the athlete's heart rate less than 65 beats per minute at precisely $t = 30$ s? Use this method to determine the heart rate after exactly 60 s.

What Is Calculus?

Two simple geometric problems originally led to the development of what is now called calculus. Both problems can be stated in terms of the graph of a function $y = f(x)$.

- **The problem of tangents:** What is the value of the slope of the tangent to the graph of a function at a given point P?

- **The problem of areas:** What is the area under a graph of a function $y = f(x)$ between $x = a$ and $x = b$?

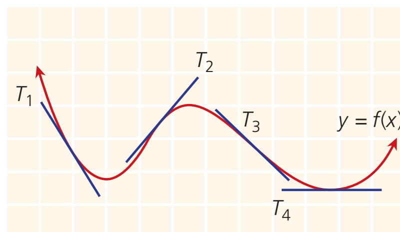


Interest in the problem of tangents and the problem of areas dates back to scientists such as Archimedes of Syracuse (287–212 B.C.), who used his vast ingenuity to solve special cases of these problems. Further progress was made in the seventeenth century, most notably by Pierre de Fermat (1601–1665) and by Isaac Barrow (1630–1677), Isaac Newton's professor at the University of Cambridge, England. Professor Barrow recognized that there was a close connection between the problem of tangents and the problem of areas. However, it took the genius of both Sir Isaac Newton (1642–1727) and Gottfried Wilhelm von Leibniz (1646–1716) to show the way to handle both problems. Using the analytic geometry of René Descartes (1596–1650), Newton and Leibniz showed independently how these two problems could be solved by means of new operations on functions, called **differentiation** and **integration.** Their discovery is considered to be one of the major advances in the history of mathematics. Further research by mathematicians from many countries using these operations has created a problem-solving tool of immense power and versatility, which is known as calculus. It is a powerful branch of mathematics, used in applied mathematics, science, engineering, and economics.

We begin our study of calculus by discussing the meaning of a tangent and the related idea of rate of change. This leads us to the study of limits and, at the end of the chapter, to the concept of the derivative of a function.

Section 3.1 — The Slope of a Tangent

You are familiar with the concept of a tangent to a curve. What geometric interpretation can be given to a tangent to the graph of a function at a point P? A tangent is the straight line that most resembles the graph near that point. Its slope tells how steep the graph is at the point of tangency. In the figure, four tangents have been drawn.

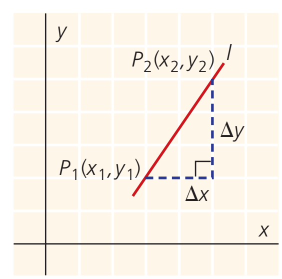


The goal of this section is to develop a method for determining the slope of a tangent at a given point on a curve. We begin with a brief review of slopes and lines.

Slopes and Lines

The slope m of the line joining points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined as

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$



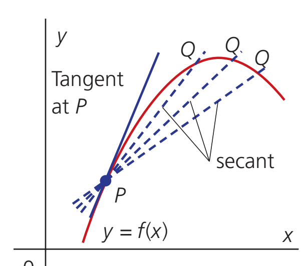
The equation of the line l in slope-point form is

$$\frac{y - y_1}{x - x_1} = m \quad \text{or} \quad y - y_1 = m(x - x_1),$$

and in slope y-intercept form is $y = mx + b$, where b is the y-intercept of the line.

To find the equation of a tangent to a curve at a given point, we first need to know the slope of the tangent. What can we do when we only have one point? We proceed as follows:

Consider a curve $y = f(x)$ and a point P that lies on the curve. Now consider another point Q on the curve. The line joining P and Q is called a **secant**. Think of Q as a moving point that slides along the curve towards P, so that the slope of the secant PQ becomes a progressively better estimate of the slope of the tangent at P.



This suggests the following definition of the slope of the tangent:

The slope of the tangent to a curve at a point P is the limiting slope of the secant PQ as the point Q slides along the curve towards P. In other words, the slope of the tangent is said to be the limit of the slope of the secant as Q approaches P along the curve.

We will illustrate this idea by finding the slope of the tangent to the parabola $y = x^2$ at $P(3, 9)$.

INVESTIGATION 1

1. Find the y-coordinates of the following points that lie on the graph of the parabola $y = x^2$.

   a. $Q_1(3.5, \quad)$ b. $Q_2(3.1, \quad)$ c. $Q_3(3.01, \quad)$ d. $Q_4(3.001, \quad)$

2. Find the slopes of the secants through $P(3, 9)$ and each of the points Q_1, Q_2, Q_3, and Q_4.

3. Find the y-coordinates of each point on the parabola and then repeat step 2 using the points.

   a. $Q_5(2.5, \quad)$ b. $Q_6(2.9, \quad)$ c. $Q_7(2.99, \quad)$ d. $Q_8(2.999, \quad)$

4. Use your results from steps 2 and 3 to estimate the slope of the tangent at point $P(3, 9)$.

5. Graph $y = x^2$ and the tangent to the graph at point $P(3, 9)$.

In this investigation, you found the slope of the tangent by finding the limiting value of the slopes of a sequence of secants. Since we are interested in points Q that are close to $P(3, 9)$ on the parabola $y = x^2$, it is convenient to write Q as $(3 + h, (3 + h)^2)$, where h is a very small non-zero number. The variable h determines the position of Q on the parabola. As Q slides along the parabola towards P, h will take on values successively smaller and closer to zero. We say that "h approaches zero" and use the notation "$h \to 0$."

INVESTIGATION 2

technology
APPENDIX P. 440

1. Using technology or graph paper, draw the parabola $f(x) = x^2$.

2. Let P be the point $(1, 1)$.

3. Find the slope of the secant through Q_1 and $P(1, 1)$, Q_2 and $P(1, 1)$, and so on, for points $Q_1(1.5, f(1.5))$, $Q_2(1.1, f(1.1))$, $Q_3(1.01, f(1.01))$, $Q_4(1.001, f(1.001))$, and $Q_5(1.0001, f(1.0001))$.

4. Graph these secants on the same utility you used in step 1.

5. Use your results to estimate the slope of the tangent to the graph of f at point P.

6. Draw the tangent at point $P(1, 1)$.

INVESTIGATION 3

1. Find the slope of the secant PQ through points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$, $h \neq 0$.

2. Explain how you could predict the slope of the tangent at point $P(3, 9)$ to the parabola $f(x) = x^2$.

The **slope of the tangent** to the parabola at point P is the limiting slope of the secant line PQ as point Q slides along the parabola; that is, as $h \to 0$. We write "$\lim\limits_{h \to 0}$" as the abbreviation for "limiting value as h approaches 0."
Therefore, from the investigation, the slope of the tangent at a point P is $\lim\limits_{h \to 0}$ (slope of the secant PQ).

EXAMPLE 1

Find the slope of the tangent to the graph of the parabola $f(x) = x^2$ at $P(3, 9)$.

Solution

Using points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$, $h \neq 0$, the slope of the secant PQ is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(3 + h)^2 - 9}{3 + h - 3} \\ &= \frac{9 + 6h + h^2 - 9}{h} \\ &= \frac{h(6 + h)}{h} \\ &= (6 + h)\end{aligned}$$

As $h \to 0$, the value of $(6 + h)$ approaches 6 and thus $\lim\limits_{h \to 0}(6 + h) = 6$.
We conclude the slope of the tangent at $P(3, 9)$ to the parabola $y = x^2$ is 6.

EXAMPLE 2

a. Use your calculator to graph the parabola $y = -\frac{1}{8}(x + 1)(x - 7)$ and plot the points on the parabola from $x = -1$ to $x = 6$, where x is an integer.

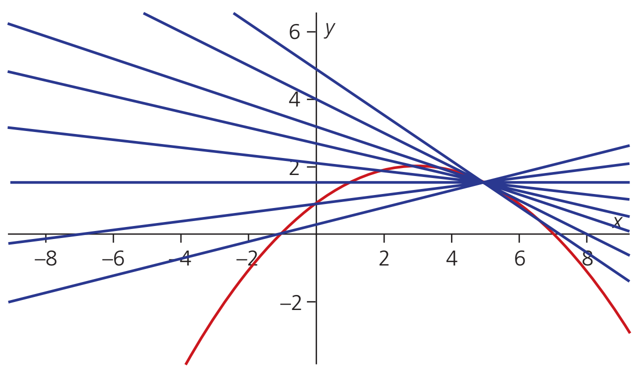
technology

b. Determine the slope of the secants using each point from part **a** and point $P(5, 1.5)$.

c. Use the result of part **b** to estimate the slope of the tangent at $P(5, 1.5)$.

Solution

a. Using the x-intercepts of -1 and 7, the equation of the axis of symmetry is $x = \frac{-1 + 7}{2} = 3$, so the x-coordinate of the vertex is 3. Substitute $x = 3$ into $y = -\frac{1}{8}(x + 1)(x - 7)$ and we get $y = -\frac{1}{8}(3 + 1)(3 - 7) = 2$.
Therefore, the vertex is (3, 2).
The y-intercept of the parabola is $\frac{7}{8}$.
The points on the parabola are $(-1, 0)$, (0, 0.875), (1, 1.5), (2, 1.875), (3, 2), (4, 1.875), (5, 1.5), and (6, 0.875).
Using graphing software, the parabola and the secants through each point and point $P(5, 1.5)$ are shown.

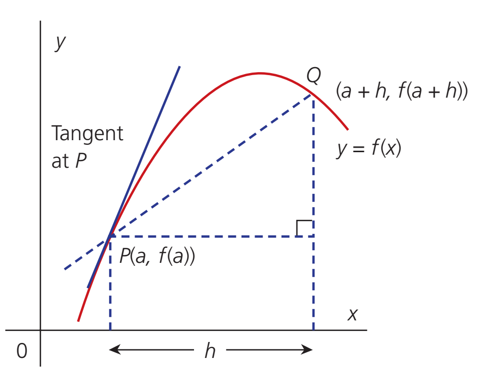


b. Using points $(-1, 0)$ and $P(5, 1.5)$, the slope is $m = \frac{1.5 - 0}{5 - (-1)} = 0.25$.
Similarly, using the other points and $P(5, 1.5)$, the slopes are 0.125, 0, -0.125, -0.25, -0.375, 0, and -0.625, respectively.

c. The slope of the tangent at $P(5, 1.5)$ is between -0.375 and -0.625. It can be determined to be -0.5 using additional points closer and closer to $P(5, 1.5)$ for values of x between 4 and 6.

The Slope of a Tangent at an Arbitrary Point

We can now generalize the method used above to derive a formula for the slope of the tangent to the graph of any function $y = f(x)$.



Let $P(a, f(a))$ be a fixed point on the graph of $y = f(x)$ and let $Q(x, y) = Q(x, f(x))$ represent any other point on the graph. If Q is a horizontal distance of h units from P, then $x = a + h$ and $y = f(a + h)$. Point Q then has coordinates $Q(a + h, f(a + h))$. The slope of the secant PQ is

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}.$$

This quotient is fundamental to calculus and is referred to as the **difference quotient.** Therefore, the slope m of the tangent at $P(a, f(a))$ is $\lim_{h \to 0}$ (slope of the secant PQ), which may be written as $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

The slope of the tangent to the graph $y = f(x)$ at point $P(a, f(a))$ is

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

EXAMPLE 3

a. Using the definition of a derivative, determine the slope of the tangent to the curve $y = -x^2 + 4x + 1$ at the point determined by $x = 3$.
b. Determine the equation of the tangent.
c. Sketch the graph of $y = -x^2 + 4x + 1$ and the tangent at $x = 3$.

Solution

a. The slope of the tangent can be determined using the expression above. In this example, $f(x) = -x^2 + 4x + 1$ and $a = 3$.

Then $\quad f(3) = -(3)^2 + 4(3) + 1 = 4$

and $\quad f(3+h) = -(3+h)^2 + 4(3+h) + 1$
$\qquad\qquad\quad = -h^2 - 2h + 4$

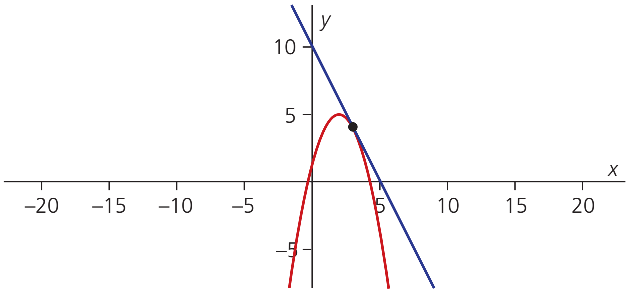
The slope of the tangent at (3, 4) is

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \to 0} \frac{-h^2 - 2h + 4 - 4}{h} \\ &= \lim_{h \to 0} \frac{h(-h-2)}{h} \\ &= \lim_{h \to 0} (-h - 2) \\ &= -2. \end{aligned}$$

The slope of the tangent at $x = 3$ is -2.

b. The equation of the tangent at (3, 4) is $\frac{y-4}{x-3} = -2$, or $y = -2x + 10$.

c. Using graphing software, we obtain



EXAMPLE 4 Determine the slope of the tangent to the rational function $f(x) = \dfrac{3x + 6}{x}$ at point (2, 6).

Solution

Using the definition, the slope of the tangent at (2, 6) is

$$\begin{aligned} m &= \lim_{h\to 0}\frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{6+3h+6}{2+h} - 6}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{12+3h-12-6h}{2+h}}{h} \\ &= \lim_{h\to 0}\frac{-3h}{2+h} \times \frac{1}{h} \\ &= \lim_{h\to 0}\frac{-3h}{h(2+h)} \\ &= \lim_{h\to 0}\frac{-3}{(2+h)} \\ &= -1.5. \end{aligned}$$

Therefore, the slope of the tangent to $f(x) = \dfrac{3x + 6}{x}$ at (2, 6) is -1.5.

EXAMPLE 5 Find the slope of the tangent to $y = f(x)$, where $f(x) = \sqrt{x}$, at $x = 9$.

Solution

At $x = 9, f(9) = \sqrt{9} = 3$.
At $x = 9 + h, f(9 + h) = \sqrt{9 + h}$.

Using the limit of the difference quotient, the slope of the tangent at $x = 9$ is

$$
\begin{aligned}
m &= \lim_{h\to 0} \frac{f(9+h)-f(9)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \times \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} \\
&= \lim_{h\to 0} \frac{(9+h)-9}{h(\sqrt{9+h}+3)} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{9+h}+3)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} \\
&= \frac{1}{6}.
\end{aligned}
$$

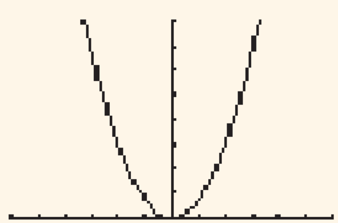
Rationalizing the numerator of a quotient often helps to simplify the calculation of some limits. When the numerator has two terms, such as $\sqrt{9+h}-3$, we multiply both the numerator and denominator by the conjugate radical; that is, by $\sqrt{9+h}+3$.

Therefore, the slope of the tangent to the function $f(x) = \sqrt{x}$ at $x = 9$ is $\frac{1}{6}$.

INVESTIGATION 4 A graphing calculator can help us "guess" the approximate value of the slope of a tangent at a point, which can then be found using the definition of the slope of the tangent, from first principles, developed in this section. For example, suppose we wish to find the slope of the tangent to $y = f(x) = x^3$ at $x = 1$.

1. Graph $Y_1 = \frac{((x+0.01)^3 - x^3)}{0.01}$.

technology



WINDOW
$X_{min} = -3$
$X_{max} = 3$
$X_{scl} = 0.5$
$Y_{min} = 0$
$Y_{max} = 8$
$Y_{scl} = 1$

2. Explain why the values for the WINDOW were chosen.
   Looking at the graph, recognize Y_1 as the difference quotient $\frac{f(a+h)-f(a)}{h}$ for $f(x) = x^3$ and $h = 0.01$. Remember that this approximates the slope of the tangent and not the graph of $f(x) = x^3$.

3. Use the TRACE function to find $X = 1.0212766$, $Y = 3.159756$. This means the slope of the secant passing through the points where $x = 1$ and $x = 1 + 0.01 = 1.01$ is about 3.2; this could be used as an approximation for the slope of the tangent at $x = 1$.

4. Can you improve this approximation? Explain how you could improve your estimate. Also, if you use different WINDOW values you can see a different-sized or differently centred graph.

5. Try once again by setting $X_{min} = -9$, $X_{max} = 10$, and note the different appearance of the graph. Use the TRACE function to find $X = 0.904\,255\,32$, $Y = 2.480\,260\,7$, then $X = 1.106\,383$, $Y = 3.705\,541\,4$. What is your guess for the slope of the tangent at $x = 1$ now? Explain why only estimation is possible.

6. Another way of using a graphing calculator to approximate the slope of the tangent is to consider h as the variable in the difference quotient. For this example, $f(x) = x^3$ at $x = 1$, look at
$$\frac{f(a+h) - f(a)}{h} = \frac{(1+h)^3 - 1^3}{h}.$$

7. Trace values of h as $h \to 0$. You can use the table or graph functions of your calculator. Graphically, we say we are looking at $\frac{(1+h)^3 - 1}{h}$ in the *neighbourhood* of $h = 0$. To do this, graph $y = \frac{(1+x)^3 - 1}{x}$ and examine the value of the function as $x \to 0$.

Exercise 3.1

Part A

1. Find the slope of the line through each pair of points.
   a. $(2, 7), (-3, -8)$
   b. $\left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, -\frac{7}{2}\right)$
   c. $(6.3, -2.6), (1.5, -1)$

2. What is the slope of a line perpendicular to the following?
   a. $y = 3x - 5$
   b. $13x - 7y - 11 = 0$

3. State the equation and sketch the graph of the following straight lines:
   a. passing through $(-4, -4)$ and $\left(-\frac{5}{3}, -\frac{5}{3}\right)$
   b. having slope 8 and y-intercept 6
   c. having x-intercept 5 and y-intercept -3
   d. passing through $(5, 6)$ and $(5, -9)$

4. Simplify each of the following:
   a. $\frac{(2+h)^2 - 4}{h}$
   b. $\frac{(5+h)^3 - 125}{h}$
   c. $\frac{(3+h)^4 - 81}{h}$
   d. $\frac{\frac{1}{1+h} - 1}{h}$
   e. $\frac{3(1+h)^2 - 3}{h}$
   f. $\frac{(2+h)^3 - 8}{h}$
   g. $\frac{\frac{3}{4+h} - \frac{3}{4}}{h}$
   h. $\frac{\frac{-1}{2+h} + \frac{1}{2}}{h}$

5. Rationalize each of the following numerators to obtain an equivalent expression.

a. $\frac{\sqrt{16+h}-4}{h}$ b. $\frac{\sqrt{h^2+5h+4}-2}{h}$ c. $\frac{\sqrt{5+h}-\sqrt{5}}{h}$

Part B

6. Find the slope m, in simplified form, of each pair of points.

a. $P(1, 3), Q(1+h, f(1+h))$ where $f(x) = 3x^2$

b. $R(1, 3), S(1+h, (1+h)^3 + 2)$

c. $T(9, 3), U(9+h, \sqrt{9+h})$

Knowledge/ Understanding

7. Consider the function $f(x) = x^3$.

a. Copy and complete the following table of values; P and Q are points on the graph of $f(x)$.

P	*Q*	Slope of Line *PQ*
(2,)	(3,)	
(2,)	(2.5,)	
(2,)	(2.1,)	
(2,)	(2.01,)	

P	*Q*	Slope of Line *PQ*
(2,)	(1,)	
(2,)	(1.5,)	
(2,)	(1.9,)	
(2,)	(1.99,)	

b. Use the results of part **a** to approximate the slope of the tangent to the graph of $f(x)$ at point P.

c. Calculate the slope of the secant PR, where the x-coordinate of R is $2 + h$.

d. Use the result of part **c** to calculate the slope of the tangent to the graph of $f(x)$ at point P.

e. Compare your answers for parts **b** and **d.**

f. Sketch the graph of $f(x)$ and the tangent to the graph at point P.

8. Find the slope of the tangent to each curve at the point whose x-value is given.

a. $y = 3x^2; (-2, 12)$ b. $y = x^2 - x$ at $x = 3$ c. $y = x^3$ at $x = -2$

9. Find the slope of the tangent to each curve at the point whose x-value is given.

a. $y = \sqrt{x-2}; (3, 1)$ b. $y = \sqrt{x-5}$ at $x = 9$ c. $y = \sqrt{5x-1}$ at $x = 2$

10. Find the slope of the tangent to each curve at the point whose x-value is given.

a. $y = \frac{8}{x}; (2, 4)$ b. $y = \frac{8}{3+x}$ at $x = 1$ c. $y = \frac{1}{x+2}$ at $x = 3$

Knowledge/Understanding

11. Find the slope of the tangent to each curve at the given point.

a. $y = x^2 - 3x$; $(2, -2)$

b. $f(x) = \frac{4}{x}$; $(-2, -2)$

c. $y = 3x^3$ at $x = 1$

d. $y = \sqrt{x - 7}$ at $x = 16$

e. $f(x) = \sqrt{16 - x}$, where $y = 5$

f. $y = \sqrt{25 - x^2}$; $(3, 4)$

g. $y = \frac{4 + x}{x - 2}$ at $x = 8$

h. $y = \frac{8}{\sqrt{x + 11}}$ at $x = 5$

12. Sketch the graph of Question 11, part **f.** Show that the slope of the tangent can be found using the properties of circles.

Communication

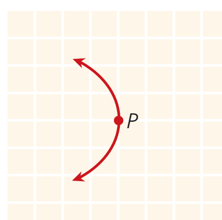
13. Explain how you would approximate the slope of the tangent at a point without using first principles.

Communication

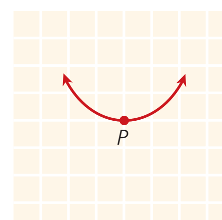
14. Sketch the graph of $y = \frac{3}{4}\sqrt{16 - x^2}$. Explain how the slope of the tangent at $P(0, 3)$ can be found without using first principles.

15. Copy the following figures. Draw an approximate tangent for each curve at point P.

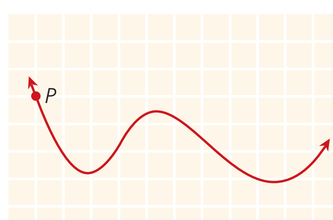
a.



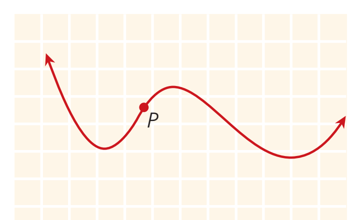
b.



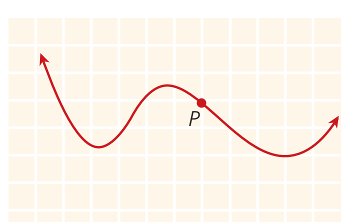
c.



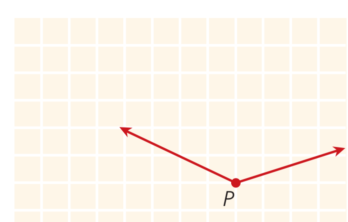
d.



e.



f.



Application **16.** Find the slope of the demand curve $D(p) = \frac{20}{\sqrt{p - 1}}$, $p > 1$, at point $(5, 10)$.

Application **17.** It is projected that t years from now, the circulation of a local newspaper will be $C(t) = 100t^2 + 400t + 5000$. Find how fast the circulation is increasing after 6 months. *Hint:* Find the slope of the tangent when t is equal to 6 months.

Thinking/Inquiry/ Problem Solving **18.** Find the coordinates of the point on the curve $f(x) = 3x^2 - 4x$, where the tangent is parallel to the line $y = 8x$.

Thinking/Inquiry/ Problem Solving **19.** Find the points on the graph of $y = \frac{1}{3}x^3 - 5x - \frac{4}{x}$ at which the slope of the tangent is horizontal.

Part C

20. Show that at the points of intersection of the quadratic functions $y = x^2$ and $y = \frac{1}{2} - x^2$, the tangents to each parabola are perpendicular.

Section 3.2 — Rates of Change

Many practical relationships involve interdependent quantities. For example, the volume of a balloon varies with its height above the ground, air temperature varies with elevation, and the surface area of a sphere varies with the length of the radius.

These and other relationships can be described by means of a function, often of the form $y = f(x)$. The **dependent variable,** y, can represent price, air temperature, area, and so forth. The **independent variable,** x, can represent time, elevation, length, and so on.

We are often interested in how rapidly the dependent variable changes when there is a change in the independent variable. This concept is called **rate of change.** In this section, we show that a rate of change can be calculated by finding the limit of a difference quotient, in the same way we find the slope of a tangent.

An object moving in a straight line is an example of a rate of change model. It is customary to use either a horizontal or a vertical line with a specified origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or down) is considered to be in the negative direction. An example of an object moving along a line would be a vehicle entering a highway and travelling north 340 km in 4 h. The average velocity would be $\frac{340}{4} = 85$ km/h, since

$$\text{average velocity} = \frac{\text{change in distance}}{\text{change in time}}.$$

If $s(t)$ gives the position of the vehicle on a straight section of the highway at time t, then the average rate of change of the position of the vehicle over a time interval is

$$\text{average velocity} = \frac{\Delta s}{\Delta t}.$$

INVESTIGATION

You are driving with a broken speedometer on a highway. At any instant you do not know how fast the car is going. Your odometer readings are the following:

t (in hours)	0	1	2	2.5	3
s(*t*) (in kilometres)	62	133	210	250	293

1. Determine the average velocity of the car over each interval.
2. The speed limit is 80 km/h. Do any of the results suggest that you were speeding at any time? If so, when?

3. Explain why there may be other times when you were travelling above the posted speed limit.
4. Compute your average velocity over the interval $4 \leq t \leq 7$ if $s(4) = 375$ km and $s(7) = 609$ km.
5. After 3 h of driving, you decide to continue driving from Goderich to Huntsville, a distance of 330 km. Using the average velocity from Question 4, how long would it take you to make this trip?

EXAMPLE 1

A pebble is dropped from a cliff with a height of 80 m. After t seconds, it is s metres above the ground, where $s(t) = 80 - 5t^2$, $0 \leq t \leq 4$.

a. Find the average velocity of the pebble between the times $t = 1$ s and $t = 3$ s.

b. Find the average velocity of the pebble between the times $t = 1$ s and $t = 1.5$ s.

c. Explain why the answers to parts **a** and **b** are different.

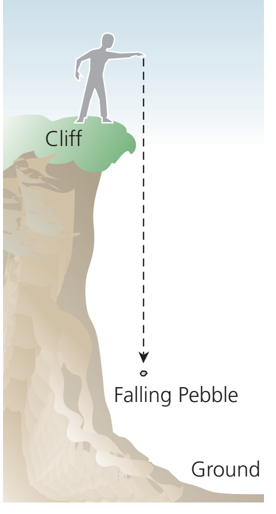
Solution

a. average velocity $= \dfrac{\Delta s}{\Delta t}$

$$s(1) = 75$$

$$s(3) = 35$$

$$\begin{aligned}\text{average velocity} &= \frac{s(3) - s(1)}{3 - 1}\\ &= \frac{35 - 75}{2}\\ &= \frac{-40}{2}\\ &= -20 \text{ m/s}\end{aligned}$$



The average velocity in this 2 s interval is -20 m/s.

b. $$\begin{aligned}s(1.5) &= 80 - 5(1.5)^2\\ &= 68.75\end{aligned}$$

$$\begin{aligned}\text{average velocity} &= \frac{s(1.5) - s(1)}{1.5 - 1}\\ &= \frac{68.75 - 75}{0.5}\\ &= -12.5 \text{ m/s}\end{aligned}$$

The average velocity in this 0.5 s interval is -12.5 m/s.

c. Since gravity causes the velocity to increase with time, the smaller interval of 0.5 s gives a lower average velocity, as well as giving a value closer to the actual velocity at time $t = 1$.

The following table shows the results of similar calculations of the average velocity over successively smaller time intervals.

Time Interval	Average Velocity (in metres per second)
$1 \le t \le 1.1$	-10.5
$1 \le t \le 1.01$	-10.05
$1 \le t \le 1.001$	-10.005

It appears that, as we shorten the time interval, the average velocity is approaching the value 10 m/s. The average velocity over the time interval $1 \le t \le 1 + h$ is

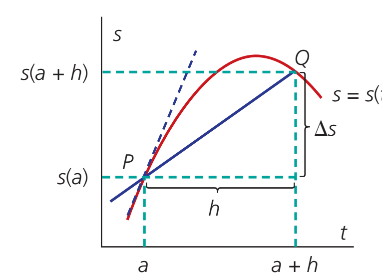
$$\begin{aligned}\text{average velocity} &= \frac{s(1+h) - s(1)}{h}\\ &= \frac{[80 - 5(+h)^2] - [80 - 5]}{h}\\ &= \frac{-(5 + 10h + 5h^2) + 5}{h}\\ &= -(10 + 5h),\ h \ne 0.\end{aligned}$$

If the time interval is very short, then h is small, so $5h$ is close to 0 and the average velocity is close to -10 m/s. The **instantaneous velocity** when $t = 1$ is defined to be the limiting value of these average values as h approaches 0. Therefore, the velocity (the word "instantaneous" is usually omitted) at time $t = 1$ s is

$$v = \lim_{h \to 0}(10 + 5h) = -10 \text{ m/s}.$$

In general, suppose that the position of an object at time t is given by the function $s(t)$. In the time interval from $t = a$ to $t = a + h$, the change in position is

$$\Delta s = s(a + h) - s(a).$$



The average velocity over this time interval is

$$\frac{\Delta s}{\Delta t} = \frac{s(a+h) - s(a)}{h}$$

which is the same as the slope of the secant PQ. The **velocity** at a particular time $t = a$ is calculated by finding the limiting value of the average velocity as $h \to 0$.

The velocity of an object, with position function $s(t)$, at time $t = a$, is

$$v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}.$$

Note that the velocity $v(a)$ is the slope of the tangent to the graph of $s(t)$ at $P(a, s(a))$.

The **speed** of an object is the absolute value of its velocity. It indicates how fast an object is moving, whereas velocity indicates both speed and direction (relative to a given coordinate system).

EXAMPLE 2 A ball is tossed straight up so that its position s, in metres, at time t, in seconds, is given by $s(t) = -5t^2 + 30t + 2$. What is the velocity of the ball at $t = 4$?

Solution

Since $s(t) = -5t^2 + 30t + 2$,

$$\begin{aligned} s(4+h) &= -5(4+h)^2 + 30(4+h) + 2 \\ &= -5h^2 - 10h + 42 \\ s(4) &= -5(4)^2 + 30(4) + 2 \\ &= 42. \end{aligned}$$

The velocity at $t = 4$ is

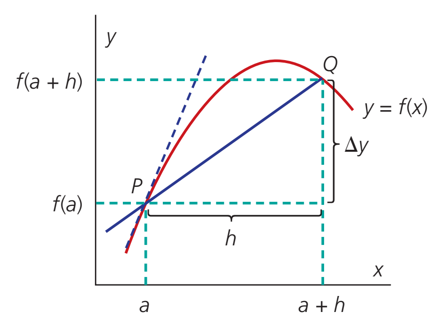
$$\begin{aligned} v(4) &= \lim_{h\to 0} \frac{s(4+h) - s(4)}{h} \\ &= \lim_{h\to 0} \frac{[-10h - 5h^2]}{h} \\ &= \lim_{h\to 0} \frac{h(-10 - 5h)}{h} \\ &= \lim_{h\to 0} (-10 - 5h) \\ &= -10. \end{aligned}$$

Therefore, the velocity of the ball is 10 m/s downwards at $t = 4$.

Velocity is only one example of the concept of **rate of change.** In general, suppose that a quantity y depends on x according to the equation $y = f(x)$. As the independent variable changes from a to $a + h$, the corresponding change in the dependent variable y is $\Delta y = f(a + h) - f(a)$.

The difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$ is called the average rate of change of y with respect to x over the interval from $x = a$ to $x = a + h$.

From the diagram, it follows that the average rate of change equals the slope of the secant PQ of the graph of $f(x)$. The rate of change of y with respect to x when $x = a$ is defined to be the limiting value of the average rate of change as $h \to 0$.



Therefore, we conclude that the rate of change of $y = f(x)$ with respect to x when $x = a$ is

$\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{h\to 0} \frac{f(a+h) - f(x)}{h}$, provided the limit exists.

It should be noted that as with velocity, the rate of change of y with respect to x at $x = a$ equals the slope of the tangent to the graph of $y = f(x)$ at $x = a$.

EXAMPLE 3 The total cost of manufacturing x units of a product is given by $C(x) = 10\sqrt{x} + \$1000$.

a. What is the total cost of manufacturing 100 items of the product?
b. What is the rate of change of the total cost with respect to the number of units, x, being produced when $x = 100$?

Solution

a. $$\begin{aligned} C(100) &= 10\sqrt{100} + 1000 \\ &= 1100 \end{aligned}$$

Therefore, the total cost of manufacturing 100 items is \$1100.

b. The rate of change of the cost at $x = 100$ is given by

$$\begin{aligned}
&\lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} \\
&= \lim_{h \to 0} \frac{10\sqrt{100 + h} + 1000 - 1100}{h} \\
&= \lim_{h \to 0} \frac{10\sqrt{100 + h} - 100}{h} \times \frac{10\sqrt{100 + h} + 100}{} \qquad \text{(Rationalizing the numerator)} \\
&= \lim_{h \to 0} \frac{100(100 + h) - 10\,000}{} \\
&= \lim_{h \to 0} \frac{100h}{} \\
&= \lim_{h \to 0} \frac{100}{} \\
&= \frac{100}{} \\
&= 0.5.
\end{aligned}$$

Therefore, the rate of change of the total cost with respect to the number of items being produced when that number is 100 is \$0.50 per item.

An Alternative Form for Finding Rates of Change

In Example 1, we determined the velocity of the pebble at $t = 1$ by taking the limit of the average velocity over the interval $1 \le t \le 1 + h$ as h approaches 0. We can also determine the velocity at $t = 1$ by considering the average velocity over the interval from 1 to a general time t and letting t approach the value 1. Then,

$$\begin{aligned}
s(1) &= 75 \\
s(t) &= 80 - 5t^2 \\
v(1) &= \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1}
\end{aligned}$$

$$= \lim_{t \to 1} \frac{5 - 5t^2}{t - 1}$$
$$= \lim_{t \to 1} \frac{5(1 - t)(1 + t)}{t - 1}$$
$$= \lim_{t \to 1} -5(1 + t)$$
$$= -10.$$

In general, the velocity of an object at time $t = a$ is $v(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$. Similarly, the rate of change of $y = f(x)$ with respect to x when $x = a$ is $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

Exercise 3.2

Part A

Knowledge/Understanding

1. The velocity of an object is given by $v(t) = t(t - 4)^2$. At what times, in seconds, is the object at rest?

Communication

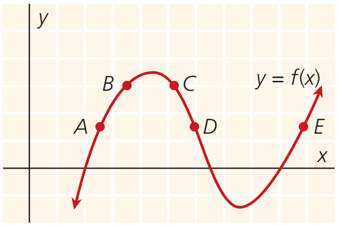
2. Give a geometrical interpretation of the following, where s is a position function.

a. $\frac{s(9) - s(2)}{7}$

b. $\lim_{h \to 0} \frac{s(6 + h) - s(6)}{h}$

3. Give a geometrical interpretation of $\lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h}$.

4. Use the graph to answer each question.



a. Between which two consecutive points is the average rate of change of the function greatest?

b. Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?

c. Sketch a tangent to the graph between points D and E such that the slope of the tangent is the same as the average rate of change of the function between B and C.

5. What is wrong with the statement “the speed of the cheetah was 65 km/h north”?

6. Is there anything wrong with the statement “a school bus had a velocity of 60 km/h for the morning run and that is why it was late on arrival”?

Part B

7. A construction worker drops a bolt while working on a high-rise building 320 m above the ground. After t seconds, the bolt has fallen a distance of s metres, where $s(t) = 320 - 5t^2$, $0 \leq t \leq 8$.
   a. Find the average velocity during the first, third, and eighth seconds.
   b. Find the average velocity for the interval $3 \leq t \leq 8$.
   c. Find the velocity at $t = 2$.

8. The function $s(t) = 8t(t + 2)$ describes the distance s, in kilometres, that a car has travelled after a time t, in hours, for $0 \leq t \leq 5$.
   a. Find the average velocity of the car during the following intervals:
      i) from $t = 3$ to $t = 4$
      ii) from $t = 3$ to $t = 3.1$
      iii) $3 \leq t \leq 3.01$
   b. Use the results of part **a** to approximate the instantaneous velocity of the car when $t = 3$.
   c. Find the velocity at $t = 3$.

Application

9. Suppose that a foreign-language student has learned $N(t) = 20t - t^2$ vocabulary terms after t hours of uninterrupted study.
   a. How many terms are learned between time $t = 2$ h and $t = 3$ h?
   b. What is the rate in terms per hour at which the student is learning at time $t = 2$ h?

Application

10. A medicine is administered to a patient. The amount, M, of the medicine, in milligrams, in 1 mL of the patient’s blood t hours after the injection is given by $M(t) = -\frac{1}{3}t^2 + t$, where $0 \leq t \leq 3$.
   a. Find the rate of change of the amount, M, 2 h after the injection.
   b. What is the significance of the fact that your answer is negative?

11. The time, t, in seconds, taken for an object dropped from a height of s metres to reach the ground is given by the formula $t = \sqrt{\frac{s}{5}}$. Determine the rate of change of the time with respect to the height when the height of an object is 125 m above the ground.

12. Suppose that the temperature, T, in degrees Celsius, varies with the height h, in kilometres, above the earth's surface according to the equation $T(h) = \frac{60}{h + 2}$. Find the rate of change of temperature with respect to height at a height of 3 km.

13. A spaceship approaching touchdown on a distant planet has height h, in metres, at time t, in seconds, given by $h = 25t^2 - 100t + 100$. When and with what speed does it land on the surface?

Application

14. A manufacturer of soccer balls finds that the profit from the sale of x balls per week is given by $P(x) = 160x - x^2$ dollars.

a. Find the profit on the sale of 40 soccer balls.

b. Find the rate of change of the profit at the production level of 40 balls per week.

technology

c. Using a graphing calculator, graph the profit function and from the graph determine for what sales levels of x the rate of change of profit is positive.

15. Use the alternate definition $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ to find the rate of change of $f(x)$ at each of the given points.

a. $f(x) = -x^2 + 2x + 3$; $(-2, -5)$

b. $f(x) = \frac{x}{x - 1}$, where $x = 2$

c. $f(x) = \sqrt{x + 1}$, where $x = 24$

Part C

Thinking/Inquiry/ Problem Solving

16. Let (a, b) be any point on the graph of $y = \frac{1}{x}$, $x > 0$. Prove that the area of the

triangle formed by the tangent through (a, b) and the coordinate axes is 2.

17. A manufacturer's total weekly cost in producing x items can be written as $C(x) = F + V(x)$, where F, a constant, represents fixed costs such as rent and utilities, and $V(x)$ represents variable costs, which depend on production level x. Show that the rate of change of the cost is independent of fixed costs.

18. A circular oil slick on the ocean spreads outward. Find the approximate rate of change of the area of the oil slick with respect to its radius when the radius is 100 m.

19. Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube.

Section 3.3 — The Limit of a Function

The notation $\lim_{x \to a} f(x) = L$ is read "the limit of $f(x)$ as x approaches a equals L" and means that the value of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a (but not equal to a). The limit $\lim_{x \to a} f(x)$ exists if and only if the limiting value from the left equals the limiting value from the right. We shall use this definition to evaluate some limits.

Note: This is an intuitive explanation of the limit of a function. A more precise definition using inequalities is important for advanced work but is not necessary for our purposes.

INVESTIGATION 1

Find the limit of $y = x^2 - 1$, as x approaches 2.

1. Copy and complete the table of values.

x	1	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5	3
$y = x^2 - 1$											

2. As x approaches 2 from the left, starting at $x = 1$, what is the approximate value of y?
3. As x approaches 2 from the right, starting at $x = 3$, what is the approximate value of y?
4. Graph $y = x^2 - 1$ using graphing software or graph paper.
5. Using arrows, illustrate that as we choose a value of x that is closer and closer to $x = 2$, the value of y gets closer and closer to a value of 3.
6. Explain why the limit of $y = x^2 - 1$ exists as x approaches 2, and give its approximate value.

INVESTIGATION 2

Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ by graphing.

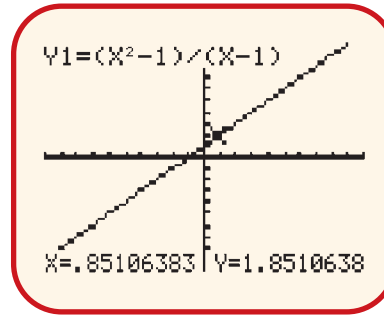
Solution

On a graphing calculator, display the graph of $f(x) = \frac{x^2 - 1}{x - 1}$, $x \neq 1$.

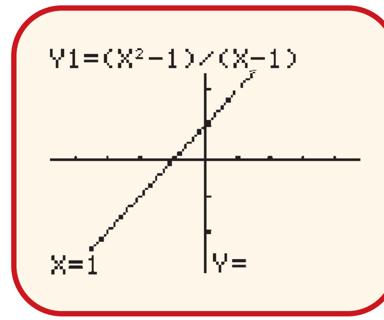
technology
APPENDIX P. 434

The graph shown on your calculator is a straight line ($f(x) = x + 1$) whereas it should be a line with the point (1, 2) deleted ($f(x) = x + 1, x \neq 1$). The WINDOW is $X_{\min} = -10$, $X_{\max} = 10$, $X_{scl} = -10$, and similarly for Y. Use the TRACE function to find $X = 0.851\,063\,83$, $Y = 1.851\,063\,8$; $X = 1.063\,829\,8$, and $Y = 2.063\,829\,8$.

Click ZOOM; select **4:ZDecimal,** ENTER.
Now, the graph of $f(x) = \frac{x^2 - 1}{x - 1}$ is displayed as a straight line with the point (1, 2) deleted. The WINDOW has new values, too.
Use the TRACE function to find $X = 0.9$, $Y = 1.9$; $X = 1$, Y has no value given; and $X = 1.1$, $Y = 2.1$.



We can estimate $\lim_{x \to 1} f(x)$. As x approaches 1 from the left, written as $x \to 1^-$, we observe that $f(x)$ approaches the value 2 from below, and as x approaches 1 from the right, written as $x \to 1^+$, $f(x)$ approaches the value 2 from above.



We say that the limit at $x = 1$ exists only if the value approached from the left is the same as the value approached from the right. From this investigation, we conclude that $\lim_{x \to 1}\left(\frac{x^2 - 1}{x - 1}\right) = 2$.

EXAMPLE 1

Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ by using a table.

Solution

We select sequences of numbers for $x \to 1^-$ and $x \to 1^+$.

x approaches 1 from the left $\rightarrow$							$\leftarrow$ x approaches 1 from the right				
x	0	0.5	0.9	0.99	0.999	1	1.001	1.01	1.1	1.5	2
$\frac{x^2 - 1}{x - 1}$	1	1.5	1.9	1.99	1.999	undefined	2.001	2.01	2.1	2.5	3
$f(x) = \frac{x^2 - 1}{x - 1}$ approaches 2 from below $\rightarrow$							$\leftarrow$ $f(x) = \frac{x^2 - 1}{x - 1}$ approaches 2 from above				

This pattern of numbers suggests $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$, as we found when graphing in Investigation 2.

EXAMPLE 2

technology
APPENDIX P. 438

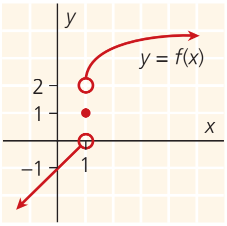
Sketch the graph of the function of

$$f(x) = \begin{cases} x - 1, \text{ if } x < 1. \\ 1, \text{ if } x = 1. \\ 2 + \sqrt{x - 1}, \text{ if } x > 1. \end{cases}$$

Determine $\lim_{x \to 1} f(x)$.

Solution

The graph of the function f consists of the line $y = x - 1$ for $x < 1$, the point (1, 1) and the square root function $y = 2 + \sqrt{x - 1}$ for $x > 1$. From the graph of $f(x)$, observe that the limit of $f(x)$ as $x \to 1$ depends on whether $x < 1$ or $x > 1$. As $x \to 1^-$, $f(x)$ approaches the value of 0, from below. We write this as $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - 1) = 0$.



Similarly, as $x \to 1^+$, $f(x)$ approaches the value 2, from above. We write this as $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2 + \sqrt{x - 1}) = 2$. (This is the same value when $x = 1$ is substituted.) These two limits are referred to as one-sided limits, because in each case only values of x on one side of $x = 1$ are considered. However, the one-sided limits are unequal—$\lim_{x \to 1^-} f(x) = 0 \neq 2 = \lim_{x \to 1^+} f(x)$—or more briefly, $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$. This implies that $f(x)$ does not approach a **single value** as $x \to 1$. We say "the limit of $f(x)$ as $x \to 1$ does not exist" and write "$\lim_{x \to 1} f(x)$ does not exist." This may be surprising, since the function $f(x)$ was defined at $x = 1$; that is, $f(1) = 1$. We can now summarize the ideas introduced in these examples.

We say that the number L is the limit of a function $y = f(x)$ as x approaches the value a, written as $\lim_{x \to a} f(x) = L$, if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$. Otherwise, $\lim_{x \to a} f(x)$ does not exist.

Exercise 3.3

Part A

1. What do you think is the appropriate limit of these sequences?
   a. 0.7, 0.72, 0.727, 0.7272, …
   b. 3, 3.1, 3.14, 3.141, 3.1415, 3.141 59, 3.141 592, …

Communication

2. Explain a process for finding a limit.

Communication

3. Write a concise description of the meaning of $\lim_{x \to 3} f(x) = 10$.

Knowledge/Understanding

4. Calculate each limit.

a. $\lim_{x\to -5} x$ b. $\lim_{x\to 3}(x+7)$ c. $\lim_{x\to 10} x^2$

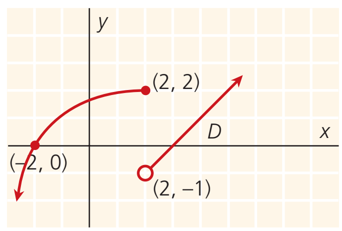
d. $\lim_{x\to -2}(4-3x^2)$ e. $\lim_{x\to 1} 4$ f. $\lim_{x\to 3} 2^x$

5. Find $\lim_{x\to 4} f(x)$, where $f(x) = \begin{cases} 1, & x \neq 4. \\ -1, & x = 4. \end{cases}$

Part B

6. For the function $y = f(x)$ in the graph below, find the following:

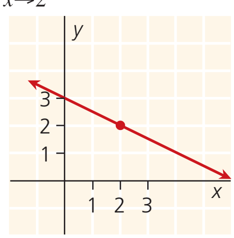
a. $\lim_{x\to -2^+} f(x)$ b. $\lim_{x\to 2^-} f(x)$ c. $\lim_{x\to 2^+} f(x)$ d. $f(2)$

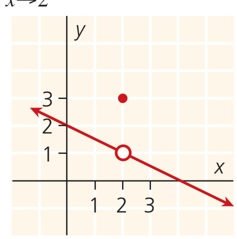


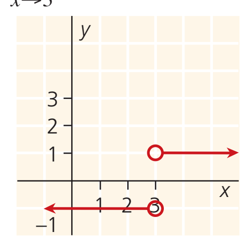
Knowledge/Understanding

7. Use the graph to find the limit, if it exists.

a. $\lim_{x\to 2} f(x)$ b. $\lim_{x\to 2} f(x)$ c. $\lim_{x\to 3} f(x)$







8. Evaluate each limit.

a. $\lim_{x\to -1}(9-x^2)$ b. $\lim_{x\to 0}\sqrt{\dfrac{x+20}{2x+5}}$ c. $\lim_{x\to 5}\sqrt{x-1}$

9. Find $\lim_{x\to 2}(x^2+1)$ and illustrate your result with a graph indicating the limiting value.

Communication

10. i) Evaluate the limits.

ii) If the limit does not exist, explain why.

a. $\lim_{x\to 0^+} x^4$ b. $\lim_{x\to 2^-}(x^2-4)$ c. $\lim_{x\to 3^-}(x^2-4)$

d. $\lim_{x\to 1^+}\dfrac{1}{x-3}$ e. $\lim_{x\to 3^+}\dfrac{1}{x+2}$ f. $\lim_{x\to 3}\dfrac{1}{x-3}$

Knowledge/Understanding

11. In each of the following, find the indicated limit if it exists. Sketch the graph of the function.

a. $f(x) = \begin{cases} x + 2, x < -1 \\ -x + 2, x \geq -1 \end{cases}$; $\lim_{x \to -1} f(x)$

b. $f(x) = \begin{cases} -x + 4, x \leq 2 \\ -2x + 6, x > 2 \end{cases}$; $\lim_{x \to 2} f(x)$

c. $f(x) = \begin{cases} 4x, x \geq \frac{1}{2} \\ \frac{1}{x}, x < \frac{1}{2} \end{cases}$; $\lim_{x \to -\frac{1}{2}} f(x)$

d. $f(x) = \begin{cases} 1, x < -0.5 \\ x^2 - 0.25, x \geq -0.5 \end{cases}$; $\lim_{x \to -0.5} f(x)$

Application

12. Sketch the graph of any function that satisfies the given conditions in each case.

a. $f(1) = 1, \lim_{x \to 1^+} f(x) = 3, \lim_{x \to 1^-} f(x) = 2$

b. $f(2) = 1, \lim_{x \to 2} f(x) = 0$

c. $f(x) = 1$, if $x < 1$ and $\lim_{x \to 1^+} f(x) = 2$

d. $f(3) = 0, \lim_{x \to 3^+} f(x) = 0$

Thinking/Inquiry/Problem Solving

13. Let $f(x) = mx + b$, where m and b are constants.
If $\lim_{x \to 1} f(x) = -2$ and $\lim_{x \to -1} f(x) = 4$, find m and b.

Part C

Thinking/Inquiry/Problem Solving

14. Determine the real values of a, b, and c for the quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, that satisfy the conditions $f(0) = 0$, $\lim_{x \to 1} f(x) = 5$, and $\lim_{x \to -2} f(x) = 8$.

15. The fish population (in thousands) in a lake at time t, in years, is modelled by the function

$$p(t) = \begin{cases} 3 + \frac{1}{12}t^2, 0 \leq t \leq 6 \\ 2 + \frac{1}{18}t^2, 6 < t \leq 12. \end{cases}$$

This function describes a sudden change in the population at time $t = 6$, due to a chemical spill.

a. Sketch the graph of $p(t)$.

b. Evaluate $\lim_{t \to 6^-} p(t)$ and $\lim_{x \to 6^+} p(t)$.

c. Determine how many fish were killed by the spill.

d. At what time did the population recover to the level before the spill?

Section 3.4 — Properties of Limits

The statement $\lim_{x \to a} f(x) = L$ says that the values of $f(x)$ become closer and closer to the number L as x gets closer and closer to the number a (from either side of a) such that $x \neq a$. This means that in finding the limit of $f(x)$ as x approaches a, there is no need to consider $x = a$. In fact, $f(a)$ need not even be defined. The only thing that matters is the behaviour of $f(x)$ near $x = a$.

EXAMPLE 1

Find $\lim_{x \to 2}(3x^2 + 4x - 1)$.

Solution

It seems clear that when x is close to 2, $3x^2$ is close to 12, and $4x$ is close to 8. Therefore, it appears that $\lim_{x \to 2}(3x^2 + 4x - 1) = 12 + 8 - 1 = 19$.

In Example 1, the limit was arrived at intuitively. It is possible to evaluate limits using the following properties of limits, which can be proved using the formal definition of limits. This is left for more advanced courses.

Properties of Limits For any real number a, suppose f and g both have limits at $x = a$.
1. $\lim_{x \to a} k = k$ for any constant k
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a}[cf(x)] = c\left(\lim_{x \to a} f(x)\right)$ for any constant c
5. $\lim_{x \to a}[f(x)g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$
6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$
7. $\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$, for n a rational number

---

EXAMPLE 2

Find $\lim_{x \to 2}(3x^2 + 4x - 1)$.

Solution

$\lim_{x \to 2}(3x^2 + 4x - 1) = \lim_{x \to 2}(3x^2) + \lim_{x \to 2}(4x) - \lim_{x \to 2}(1)$

$$
\begin{aligned}
&= 3\lim_{x\to 2}(x^2) + 4\lim_{x\to 2}(x) - 1 \\
&= 3\Big[\lim_{x\to 2} x\Big]^2 + 4(2) - 1 \\
&= 3(2)^2 + 8 - 1 \\
&= 19
\end{aligned}
$$

> *Note:* If f is a polynomial function, then $\lim_{x\to a} f(x) = f(a)$.

EXAMPLE 3 Evaluate $\lim_{x\to -1} \dfrac{x^2 - 5x + 2}{2x^3 + 3x + 1}$.

Solution

$$
\begin{aligned}
\lim_{x\to -1} \frac{x^2 - 5x + 2}{2x^3 + 3x + 1} &= \frac{\lim_{x\to -1} (x^2 - 5x + 2)}{\lim_{x\to -1} (2x^3 + 3x + 1)} \\
&= \frac{(-1)^2 - 5(-1) + 2}{} \\
&= \frac{8}{-4} \\
&= -2
\end{aligned}
$$

EXAMPLE 4 Evaluate $\lim_{x\to 5} \sqrt{\dfrac{x^2}{x - 1}}$.

Solution

$$
\begin{aligned}
\lim_{x\to 5} \sqrt{\frac{x^2}{x - 1}} &= \sqrt{\lim_{x\to 5} \frac{x^2}{x - 1}} \\
&= \sqrt{\frac{\lim_{x\to 5} x^2}{\lim_{x\to 5} (x - 1)}} \\
&= \sqrt{\frac{25}{4}} \\
&= \frac{5}{2}
\end{aligned}
$$

Sometimes the limit of $f(x)$ as x approaches a cannot be found by direct substitution. This is of special interest when direct substitution results in an **indeterminate form** $\left(\frac{0}{0}\right)$. In such cases, we look for an equivalent function that agrees with f for all values except the troublesome value $x = a$. Here are some examples.

EXAMPLE 5

Find $\lim\limits_{x \to 3} \dfrac{x^2 - 2x - 3}{x - 3}$.

Solution

If we try substitution, we obtain $\frac{0}{0}$, an indeterminate form. The next step is to simplify the function by factoring and reducing to see if the limit of the reduced form can be evaluated.

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} \frac{(x + 1)(x - 3)}{x - 3} = \lim_{x \to 3} (x + 1)$$

This reduction is valid only if $x \neq 3$. This is not a problem, since $\lim\limits_{x \to 3}$ is concerned with values as x approaches 3, not the value $x = 3$. Therefore,

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} (x + 1) = 4.$$

EXAMPLE 6

A useful technique for finding limits is to rationalize either the numerator or the denominator to obtain an algebraic form that is not indeterminate.

Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x}$.

Solution

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} &= \lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} \times \frac{\sqrt{x + 1} + 1}{\qquad\qquad} \\
&= \lim_{x \to 0} \frac{x + 1 - 1}{\qquad\qquad} \\
&= \lim_{x \to 0} \frac{x}{\qquad\qquad} \\
&= \lim_{x \to 0} \frac{1}{\qquad\qquad} \\
&= \frac{1}{2}
\end{aligned}
$$

(Multiplying by $\dfrac{\sqrt{x + 1} + 1}{\qquad\qquad}$)

INVESTIGATION

Here is an alternate technique for finding the value of a limit.

1. Find $\lim\limits_{x \to 1} \dfrac{(x - 1)}{\qquad\qquad}$ by rationalizing.

2. Let $u = \sqrt{x}$, and rewrite $\lim\limits_{x \to 1} \dfrac{(x - 1)}{\qquad\qquad}$ in terms of u. Since $x = u^2$, and $\sqrt{x} \geq 0$, and $u \geq 0$, it follows as x approaches the value of 1, u approaches the value of 1. Use this substitution to find $\lim\limits_{u \to 1} \dfrac{(u^2 - 1)}{u - 1}$ by reducing the rational expression.

EXAMPLE 7 Evaluate $\lim\limits_{x\to 0}\dfrac{(x+8)^{\frac{1}{3}}-2}{}$.

Solution

This quotient is indeterminate $\left(\frac{0}{0}\right)$ when $x = 0$. Rationalizing the term $(x + 8)^{\frac{1}{3}}$ is not so easy. However, the expression can be simplified by substitution.

Let $u = (x + 8)^{\frac{1}{3}}$. Then $u^3 = x + 8$ and $x = u^3 - 8$. As x approaches the value 0, u approaches the value 2. The given limit becomes

$$\begin{aligned}\lim_{x\to 0}\frac{(x+8)^{\frac{1}{3}}-2}{} &= \lim_{u\to 2}\frac{u-2}{u^3-8}\\ &= \lim_{u\to 2}\frac{u-2}{}\\ &= \lim_{u\to 2}\frac{1}{u^2+2u+4}\\ &= \frac{1}{12}.\end{aligned}$$

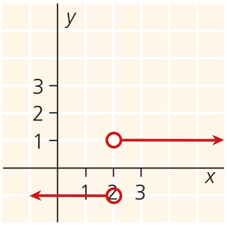
EXAMPLE 8 Evaluate $\lim\limits_{x\to 2}\dfrac{|x-2|}{}$. Illustrate with a graph.

Solution

Consider

$$f(x) = \frac{|x-2|}{} = \begin{cases}\dfrac{x-2}{x-2}, \text{ if } x > 2\\[2mm] \dfrac{-(x-2)}{x-2}, \text{ if } x < 2\end{cases}$$

$$= \begin{cases}1, \text{ if } x > 2.\\ -1, \text{ if } x < 2.\end{cases}$$



Notice that $f(2)$ is not defined and that we must consider left- and right-hand limits.

$$\lim_{x\to 2^{-}1} f(x) = \lim_{x\to 2^{-}1}(-1) = -1$$

$$\lim_{x\to 2^{+}1} f(x) = \lim_{x\to 2^{+}1}(1) = 1$$

Since the left- and right-hand limits are not the same, we conclude that

$\lim\limits_{x\to 2}\dfrac{|x-2|}{}$ does not exist.

EXAMPLE 9

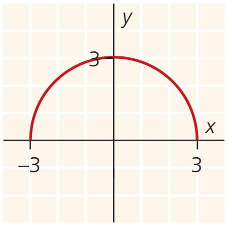
a. Evaluate $\lim\limits_{x\to 3^-} \sqrt{9 - x^2}$.

b. Explain why the limit as x approaches 3^+ cannot be determined.

c. What can you conclude about the $\lim\limits_{x\to 3} \sqrt{9 - x^2}$?

Solution

a. The graph of $f(x) = \sqrt{9 - x^2}$ is the semicircle $y = \sqrt{9 - x^2}$ as illustrated below.



From the graph, the left-hand limit at $x = 3$ is 0. Therefore, $\lim\limits_{x\to 3^-} \sqrt{9 - x^2} = 0$.

b. The function is not defined for $x > 3$.

c. $\lim\limits_{x\to 3} \sqrt{9 - x^2}$ does not exist because the function is not defined on both sides of 3.

In this section, we have learned the properties of limits and developed algebraic methods for evaluating limits. The examples in this section have complemented the table of values and graphing techniques introduced in previous sections. Five techniques for evaluating simple limits that have indeterminate quotients were illustrated:

- direct substitution
- factoring
- rationalizing
- change of variable
- one-sided limits

In each case, a graph can be utilized to check your result.

Exercise 3.4

Part A

1. Is there a different value for the answers among $\lim_{x\to 2}(3 + x)$, $\lim_{x\to 2} 3 + x$, and $\lim_{x\to 2}(x + 3)$?

Communication

2. How do you find the limit of a rational function?

Communication

3. Once you know the $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

4. Evaluate each limit.

a. $\lim_{x\to 2}\frac{3x}{x^2 + 2}$ b. $\lim_{x\to -1}(x^4 + x^3 + x^2)$ c. $\lim_{x\to 9}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

d. $\lim_{x\to 2\pi}(x^3 + \pi^2 x - 5\pi^3)$ e. $\lim_{x\to 0}\left(\sqrt{3 + \sqrt{1 + x}}\right)$ f. $\lim_{x\to -3}\sqrt{\frac{x - 3}{2x + 4}}$

Part B

Knowledge/Understanding

technology

5. Use a graphing calculator to graph the function and to estimate the limit. Then find the limit by substitution.

a. $\lim_{x\to -2}\frac{x^3}{x - 2}$ b. $\lim_{x\to 1}\frac{2x}{\quad}$

6. Show that $\lim_{t\to 1}\frac{t^3 - t^2 - 5t}{6 - t^2} = -1$.

Knowledge/Understanding

7. Evaluate the limit of each indeterminate quotient.

a. $\lim_{x\to 2}\frac{4 - x^2}{2 - x}$ b. $\lim_{x\to -2}\frac{4 - x^2}{2 + x}$ c. $\lim_{x\to 0}\frac{7x - x^2}{x}$

d. $\lim_{x\to -1}\frac{2x^2 + 5x + 3}{x + 1}$ e. $\lim_{x\to -\frac{4}{3}}\frac{3x^2 + x - 4}{3x + 4}$ f. $\lim_{x\to 3}\frac{x^3 - 27}{x - 3}$

g. $\lim_{x\to -2}\frac{x^3 + 2x^2 - 4x - 8}{x + 2}$ h. $\lim_{x\to 2}\frac{2x^3 - 5x^2 + 3x - 2}{\quad}$ i. $\lim_{x\to 0}\frac{\sqrt{x + 1} - 1}{x}$

j. $\lim_{x\to 0}\frac{2 - \sqrt{4 + x}}{x}$ k. $\lim_{x\to 4}\frac{\sqrt{x} - 2}{\quad}$ l. $\lim_{x\to 0}\frac{\sqrt{7 - x} - \sqrt{7 + x}}{x}$

m. $\lim_{x\to 1}\frac{\sqrt{5 - x} - \sqrt{3 + x}}{\quad}$ n. $\lim_{x\to 4}\frac{2 - \sqrt{x}}{\quad}$ o. $\lim_{x\to 0}\frac{2^{2x} - 2^x}{2^x - 1}$

8. Evaluate the limit by change of variable.

a. $\lim_{x\to 8}\frac{\sqrt[3]{x} - 2}{\quad}$ b. $\lim_{x\to 27}\frac{27 - x}{x^{\frac{1}{3}} - 3}$ c. $\lim_{x\to 1}\frac{x^{\frac{1}{6}} - 1}{\quad}$

d. $\lim_{x\to 1}\frac{x^{\frac{1}{6}} - 1}{x^{\frac{1}{3}} - 1}$ e. $\lim_{x\to 4}\frac{\sqrt{x} - 2}{\quad}$ f. $\lim_{x\to 0}\frac{(x + 8)^{\frac{1}{3}} - 2}{x}$

9. Evaluate each limit, if it exists, using any appropriate technique.

a. $\lim\limits_{x\to 4}\dfrac{16-x^2}{x^3+64}$ b. $\lim\limits_{x\to 4}\dfrac{x^2-16}{x^2-5x+6}$ c. $\lim\limits_{x\to 1}\dfrac{x^3+x^2-5x+3}{x^2-2x+1}$

d. $\lim\limits_{x\to -1}\dfrac{x^2+x}{x+1}$ e. $\lim\limits_{x\to 6^+}\dfrac{\sqrt{x^2-5x-6}}{}$ f. $\lim\limits_{x\to 0}\dfrac{(2x+1)^{\frac{1}{3}}-1}{x}$

g. $\lim\limits_{x\to 2}\dfrac{x^2-4}{\left(\frac{1}{x}\right)-\frac{1}{2}}$ h. $\lim\limits_{x\to 3}\dfrac{\sqrt{x+1}-2}{x-3}$ i. $\lim\limits_{x\to 0}\dfrac{x^2-9x}{5x^3+6x}$

j. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{}$ k. $\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

l. $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}\right)\left(\dfrac{1}{x+3}-\dfrac{2}{3x+5}\right)$

10. By using one-sided limits, determine whether the limit exists. Illustrate the results geometrically by sketching the graph of each function.

a. $\lim\limits_{x\to 5}\dfrac{|x-5|}{x-5}$ b. $\lim\dfrac{|2x-5|(x+1)}{2x-5}$

c. $\lim\limits_{x\to 2}\dfrac{x^2-x-2}{|x-2|}$ d. $\lim\limits_{x\to -2}\dfrac{(x+2)^3}{|x+2|}$

Application

11. **Charles' Law and Absolute Zero** Jacques Charles (1746–1823) discovered that the volume of gas at a constant pressure increases linearly with the temperature of the gas. In the table, one mole of hydrogen is held at a constant pressure of one atmosphere. The volume V is measured in litres and the temperature T is measured in degrees Celsius.

T	−40	−20	0	20	40	60	80
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

a. By finding a difference row, show that T and V are related by a linear relation.

b. Find the linear equation V in terms of T.

c. Solve for T in terms of V for the equation in part **b.**

d. Show that $\lim\limits_{V\to 0^+} T$ is approximately -273.15. *Note:* This represents the approximate number of degrees on the Celsius scale of absolute zero on the Kelvin scale (0 K).

e. Using the information found in parts **b** and **d,** draw a graph of V versus T.

Thinking/Inquiry/ Problem Solving

12. Show, using the properties of limits, that if
$\lim_{x\to5} f(x) = 3$, then $\lim_{x\to5} \frac{x^2-4}{f(x)} = 7$.

13. If $\lim_{x\to4} f(x) = 3$, use the properties of limits to evaluate each limit.

a. $\lim_{x\to4} [f(x)]^3$ b. $\lim_{x\to4} \frac{[f(x)]^2 - x^2}{f(x) + x}$ c. $\lim_{x\to4} \sqrt{3f(x) - 2x}$

Part C

14. If $\lim_{x\to0} \frac{f(x)}{x} = 1$ and $g(0) \neq 0$, then evaluate each limit.

a. $\lim_{x\to0} f(x)$ b. $\lim_{x\to0} \frac{f(x)}{g(x)}$

15. If $\lim_{x\to0} \frac{f(x)}{x} = 1$ and $\lim_{x\to0} \frac{g(x)}{x} = 2$, then evaluate each limit.

a. $\lim_{x\to0} f(x)$ b. $\lim_{x\to0} g(x)$ c. $\lim_{x\to0} \frac{f(x)}{g(x)}$

16. Evaluate $\lim_{x\to0} \frac{\sqrt{x+1} - \sqrt{2x+1}}{}$ of the indeterminate quotient.

17. Does $\lim_{x\to1} \frac{x^2 + |x-1| - 1}{|x-1|}$ exist? Illustrate your result by sketching a graph of the function.

18. For what value of b does $\lim_{x\to1} \frac{x^2 + bx - 3}{x - 1}$ exist?

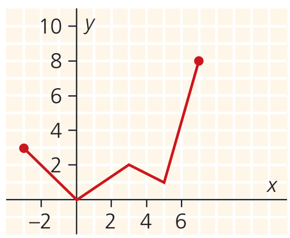
19. For what values of m and b is the statement $\lim_{x\to0} \frac{\sqrt{mx+b} - 3}{x} = 1$?

Section 3.5 — Continuity

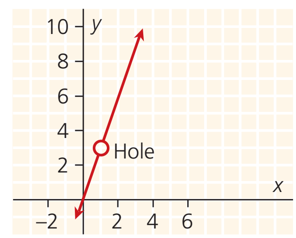
The idea of **continuity** may be thought of informally as the idea of being connected to one's neighbours. The concept arose from the notion of a graph "without breaks or jumps or gaps."

When we talk about a function being *continuous at a point*, we mean that the graph passes through the point without a break. A graph that is *not continuous at a point* (sometimes referred to as being *discontinuous at a point*) has a break of some type at the point. The following graphs illustrate these ideas.

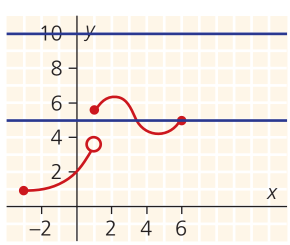
a. Continuous for all values of the domain



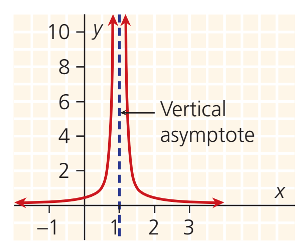
b. Discontinuous at $x = 1$



c. Discontinuous at $x = 1$



d. Discontinuous at $x = 1$



What conditions must be satisfied for a function f to be continuous at a? First, $f(a)$ must be defined. The curves in figure **b** and figure **d** are not continuous at $x = 1$ because they are not defined at $x = 1$.

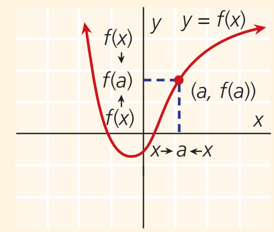
A second condition for continuity at a point $x = a$ is that the function makes no jumps there. This means that if "x is close to a," then $f(x)$ must be close to $f(a)$. This condition is satisfied if $\lim_{x \to a} f(x) = f(a)$. Looking at the graph in figure **c**, we see that $\lim_{x \to 1} f(x)$ does not exist and the function is therefore not continuous at $x = 1$.

We can now define the **continuity of a function at a point.**

The function $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and if

$$\lim_{x \to a} f(x) = f(a).$$

Otherwise, $f(x)$ is discontinuous at $x = a$.



The geometrical meaning of f being continuous at $x = a$ is that as $x \to a$, the points $(x, f(x))$ on the graph of f converge at the point $(a, f(a))$ ensuring the graph of f is unbroken at $(a, f(a))$.

EXAMPLE 1

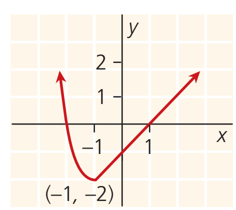
a. Graph the function $f(x) = \begin{cases} x^2 - 3, & x \le -1 \\ x - 1, & x > -1. \end{cases}$

b. Find $\lim_{x \to -1} f(x)$.

c. Find $f(-1)$.

Solution

a.



b. From the graph $\lim_{x \to -1} f(x) = -2$. *Note:* Both the left-hand and right-hand limits are equal.

c. $f(-1) = -2$
Therefore, $f(x)$ is continuous at $x = -1$.

EXAMPLE 2

Test the continuity of each of the following functions at $x = 2$. If a function is not continuous at $x = 2$, give a reason why it is not continuous.

a. $f(x) = x^3 - x$

b. $g(x) = \dfrac{x^2 - x - 2}{x - 2}$

c. $h(x) = \dfrac{x^2 - x - 2}{x - 2}$, if $x \ne 2$ and $h(2) = 3$

d. $F(x) = \dfrac{1}{(x - 2)^2}$

e. $G(x) = \begin{cases} 4 - x^2, & \text{if } x < 2 \\ 3, & \text{if } x \ge 2 \end{cases}$

Solution

a. The function f is continuous at $x = 2$ since $f(2) = 6 = \lim_{x \to 2} f(x)$. (Polynomial functions are continuous at all real values of x.)

b. The function g is not continuous at $x = 2$ because g is not defined at this point.

c. Since
$$\begin{aligned}\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} &= \lim_{x \to 2} \frac{(x - 2)(x + 1)}{(x - 2)} \\ &= \lim_{x \to 2} (x + 1) \\ &= 3 \\ &= h(2),\end{aligned}$$
therefore, $h(x)$ is continous at $x = 2$.

d. The function F is not continuous at $x = 2$ because $F(2)$ is not defined.

e. Since
$$\begin{aligned}\lim_{x \to 2^-} G(x) &= \lim_{x \to 2^-} (4 - x^2) \\ &= 0 \\ \text{and } \lim_{x \to 2^+} G(x) &= \lim_{x \to 2^+} (3) \\ &= 3,\end{aligned}$$
therefore, $\lim_{x \to 2} G(x)$ does not exist, since the function is not continuous at $x = 2$.

INVESTIGATION

technology

To test the definition of continuity by graphing, investigate the following:

1. Draw the graph for each function in Example 2.
2. Which of the graphs are continuous, contain a hole or a jump, or have a vertical asymptote?
3. Given only the defining sentence of a function $y = f(x)$ such as $f(x) = \frac{8x^3 - 9x + 5}{x^2 + 300x}$, explain why the graphing technique to test for continuity on an interval may be less suitable.
4. Find where $f(x) = \frac{8x^3 - 9x + 5}{x^2 + 300x}$ is not defined and the intervals where it is continuous.

Exercise 3.5

Part A

Communication **1.** How can looking at a graph of a function help you tell where the function is continuous?

Communication **2.** What does it mean for a function to be continuous over a given domain?

Knowledge/Understanding

3. What are the basic types of discontinuity? Give an example of each.

4. Find the value(s) of x at which the functions are discontinuous.

a. $f(x) = \frac{9 - x^2}{x - 3}$ b. $g(x) = \frac{7x - 4}{x}$ c. $h(x) = \frac{x^2 + 1}{x^3}$

d. $f(x) = \frac{x - 4}{x^2 - 9}$ e. $g(x) = \frac{13x}{x^2 + x - 6}$ f. $h(x) = \begin{cases} -x, x \leq 3 \\ 1 - x, x > 3 \end{cases}$

Part B

Knowledge/Understanding

5. Find all values of x for which the given functions are continuous.

a. $f(x) = 3x^5 + 2x^3 - x$ b. $g(x) = \pi x^2 - 4.2x + 7$

c. $h(x) = \frac{x^2 + 16}{x^2 - 5x}$ d. $f(x) = \sqrt{x + 2}$

e. $g(x) = 10^x$ f. $h(x) = \frac{16}{x^2 + 25}$

6. Examine the continuity of $g(x) = x + 3$ at the point $x = 2$.

7. Sketch a graph of $h(x) = \begin{cases} x - 1, x < 3 \\ 5 - x, x \geq 3 \end{cases}$

and determine if the function is continuous everywhere.

Knowledge/Understanding

8. Sketch a graph of $f(x) = \begin{cases} x^2, x < 0. \\ 3, x \geq 0. \end{cases}$ Is the function continuous?

Application

9. Recent postal rates for letter mail within Canada for non-standard and oversized items are given in the following table. Maximum dimensions for oversized letter mail are 380 mm × 270 mm × 20 mm.

100 g or Less	Between 100 g and 200 g	Between 200 g and 500 g
\$0.92	\$1.50	\$2.00

Draw a graph of the cost in dollars of mailing a non-standard envelope as a function of its mass in grams. Where are the discontinuities of this function?

Knowledge/Understanding

10. Determine whether $f(x) = \frac{x^2 - x - 6}{x - 3}$ is continuous at $x = 3$.

11. Examine the continuity of the function $f(x) = \begin{cases} x, \text{ if } x \leq 1. \\ 1, \text{ if } 1 < x \leq 2. \\ 3, \text{ if } x > 2. \end{cases}$

12. $g(x) = \begin{cases} x + 3, x \neq 3. \\ 2 + \sqrt{k}, x = 3. \end{cases}$ Find k, if $g(x)$ is continuous.

Part C

Thinking/Inquiry/ Problem Solving

13. Find constants a and b such that the function

$$f(x) = \begin{cases} -x, & \text{if } -3 \leq x \leq -2 \\ ax^2 + b, & \text{if } -2 < x < 0 \\ 6, & \text{if } x = 0 \end{cases}$$

is continuous for $-3 \leq x \leq 0$.

14. Consider the function $g(x) = \begin{cases} \dfrac{x|x-1|}{x-1}, & \text{if } x \neq 1. \\ 0, & \text{if } x = 1. \end{cases}$

a. Evaluate $\lim\limits_{x \to 1^+} g(x)$ and $\lim\limits_{x \to 1^-} g(x)$, then determine whether $\lim\limits_{x \to 1} g(x)$ exists.

b. Sketch the graph of $g(x)$ and identify any points of discontinuity.

Key Concepts Review

We began our introduction to calculus by considering the slope of a tangent and the related idea of rate of change. This led us to the study of limits and laid the groundwork for Chapter 4 and the concept of the derivative of a function.

Consider the following brief summary to confirm your understanding of key concepts covered in Chapter 3.

- Slope of the tangent is the limit of the slope of the secant, as Q approaches P along the curve
- Slope of a tangent at an arbitrary point
- Rates of change, average velocity, and velocity
- The Limit of a Function exists when the limiting value from the left equals the limiting value from the right
- Properties of Limits and indeterminate forms $\frac{0}{0}$
- Continuity is described as a graph "without breaks or jumps or gaps"

Formulas

- The slope of the tangent to the graph $y = f(x)$ at point $P(a, f(a))$ is
  $m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.
- average velocity $= \frac{\text{change in distance}}{\text{change in time}}$
- The velocity (instantaneous) of an object, with position function $s(t)$, at time $t = a$ is
  $v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}$.
- If f is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.
- The function $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and if $\lim_{x \to a} f(x) = f(a)$.

Career Link wrap-up investigate and apply

CHAPTER 3: ASSESSING ATHLETIC PERFORMANCE

An Olympic coach has developed a six-minute fitness test for her team members that sets target values for heart rates. The monitor they have available counts the total number of heartbeats starting from a rest position at "time zero." Results for one of the team members are as follows:

Time (in minutes)	Number of Beats
0.0	0
1.0	55
2.0	120
3.0	195
4.0	280
5.0	375
6.0	480

a. The coach has established that each athlete's heart rate must not exceed 100 beats per minute at exactly 3 min. Using a graphical technique, determine if this athlete meets the coach's criteria.

b. The coach also needs to know the instant in time when an athlete's heart rate actually exceeds 100 beats per minute. Explain how you would solve this problem graphically. Would this be an efficient method? Explain. How is this question different from part **a**?

c. Build a mathematical model with the total number of heartbeats as a function of time ($n = f(t)$) by first determining the degree of the polynomial then using the graphing calculator to obtain an algebraic model.

d. Solve **b** algebraically by obtaining an expression for the instantaneous rate of change of number of heartbeats, heart rate, as a function of time ($r = f(t)$) using the methods presented in this chapter. Compare the accuracy and efficiency of solving this question graphically and algebraically. ●

Review Exercise

1. Consider the graph of the function $f(x) = 5x^2 - 8x$.
   a. Find the slope of the secant that joins the points on the graph given by $x = -2$ and $x = 3$.
   b. Determine the average rate of change as x changes from -1 to 4.
   c. Find an equation for the line tangent to the graph of the function at $x = 1$.

2. Find the slope of the tangent to the given function at the given point.
   a. $f(x) = \frac{3}{x + 1}$, $P(2, 1)$
   b. $g(x) = \sqrt{x + 2}$, $x = -1$
   c. $h(x) = \frac{2}{\quad}$, $x = 4$
   d. $f(x) = \frac{5}{x - 2}$, $x = 4$

3. Find the slope of the graph of $f(x) = \begin{cases} 4 - x^2, x \leq 1 \\ 2x + 1, x > 1 \end{cases}$ at each of the following points:
   a. $P(-1, 3)$
   b. $P(2, 5)$

4. The height (in metres) that an object has fallen from a height of 180 m is given by the position function $s(t) = -5t^2 + 180$, where $t \geq 0$ and t is in seconds.
   a. Find the average velocity during each of the first two seconds.
   b. Find the velocity of the object when $t = 4$.
   c. At what velocity will the object hit the ground?

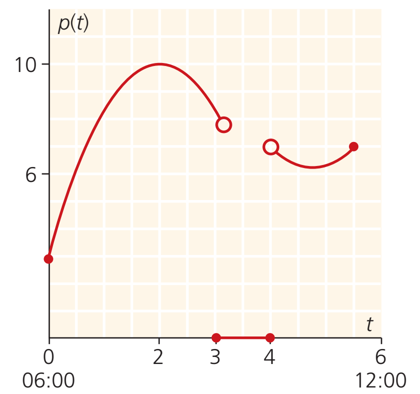
5. After t minutes of growth, a certain bacterial culture has a mass in grams of $M(t) = t^2$.
   a. How much does it grow during the time $3 \leq t \leq 3.01$?
   b. What is its average rate of growth during the time interval $3 \leq t \leq 3.01$?
   c. What is its rate of growth when $t = 3$?

6. It is estimated that t years from now, the amount of waste accumulated, Q, in tonnes, will be $Q(t) = 10^4(t^2 + 15t + 70)$ tonnes, $0 \leq t \leq 10$.
   a. How much waste has been accumulated up to now?
   b. What will be the average rate of change of this quantity over the next three years?

c. What is the present rate of change of this quantity?

d. When will the rate of change reach 3.0×10^5 tonnes per year?

7. The electrical power $p(t)$, in kilowatts, being used by a household as a function of time t, in hours, is modelled by the graph, where $t = 0$ corresponds to 06:00.



The graph indicates peak use at 08:00 and a power failure between 09:00 and 10:00.

a. Find $\lim_{t \to 2} p(t)$.

b. Determine $\lim_{t \to 4^+} p(t)$ and $\lim_{t \to 4^-} p(t)$.

c. For what values of t is $p(t)$ discontinuous?

8. Sketch a graph of any function that satisfies the given conditions in each case.

a. $\lim_{x \to -1} f(x) = 0.5$ and f is discontinuous at $x = -1$.

b. $f(x) = -4$ if $x < 3$, f is an increasing function when $x > 3$, $\lim_{x \to 3^+} f(x) = 1$.

9. Sketch the graph of the function $f(x) = \begin{cases} x + 1, x < -1. \\ -x + 1, -1 \le x < 1. \\ x - 2, x > 1. \end{cases}$

a. Find all values at which the function is discontinuous.

b. Find the limits at those values, if they exist.

10. Determine whether $f(x) = \dfrac{x^2 - x - 6}{x - 3}$ is continuous at $x = 3$.

11. Consider the function $f(x) = \dfrac{2x - 2}{x^2 + x - 2}$.

a. For what values of x is f discontinuous?

b. At each point where f is discontinuous, determine the limit of $f(x)$, if it exists.

technology

12. Use a graphing calculator to graph the function and estimate the limits, if they exist.

a. $f(x) = \dfrac{1}{x^2}$, $\lim_{x \to 0} f(x)$

b. $g(x) = x(x - 5)$, $\lim_{x \to 0} g(x)$

c. $h(x) = \dfrac{x^3 - 27}{x^2 - 9}$, $\lim_{x \to 4} h(x)$ and $\lim_{x \to -3} h(x)$

13. Complete each table and use the result to estimate the limit. Use a graphing calculator to graph the function to confirm your result.

a. $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - x - 2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

b. $\lim\limits_{x \to 1} \dfrac{x - 1}{x^2 - 1}$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$						

14. Complete the table and use the results to estimate the limit. Then determine the limit using an algebraic technique and compare the answer with the estimate.

$\lim\limits_{x \to 0} \dfrac{\sqrt{x + 3} - \sqrt{3}}{}$

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

15 a. Complete the table to approximate the limit of $f(x) = \dfrac{\sqrt{x + 2} - 2}{}$ as $x \to 2$.

x	2.1	2.01	2.001	2.0001
$f(x) = \dfrac{\sqrt{x+2} - 2}{x - 2}$				

b. Use a graphing calculator to graph f and use the graph to approximate the limit.

c. Use the technique of rationalizing the numerator to find the $\lim\limits_{x \to 2} \dfrac{\sqrt{x + 2} - 2}{}$.

16. Evaluate the limit of each difference quotient. In each case, interpret the limit as the slope of the tangent to a curve at a specific point.

a. $\lim\limits_{h \to 0} \dfrac{(5 + h)^2 - 25}{h}$

b. $\lim\limits_{h \to 0} \dfrac{\sqrt{4 + h} - 2}{h}$

c. $\lim\limits_{h \to 0} \dfrac{\frac{1}{(4 + h)} - \frac{1}{4}}{}$

d. $\lim\limits_{h \to 0} \dfrac{(343 + h)^{\frac{1}{3}} - 7}{h}$

17. Evaluate each limit using one of the algebraic methods discussed in the text, if the limit exists.

a. $\lim\limits_{x \to -2} \dfrac{x^2 - 7}{x^2 + x}$

b. $\lim\limits_{x \to a} (5x^2 - 3x + 7)$

c. $\lim\limits_{x \to -6} \dfrac{1}{6 + x}$

d. $\lim\limits_{x\to 0} 10x$

e. $\lim\limits_{x\to -6} \dfrac{x^2-36}{x+6}$

f. $\lim\limits_{x\to -4} \dfrac{x^2+12x+32}{x+4}$

g. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^3-8}$

h. $\lim\limits_{x\to a} \dfrac{(x+4a)^2-25a^2}{x-a}$

i. $\lim\limits_{x\to 0} \dfrac{x^2+3x}{3x^2-7x}$

j. $\lim\limits_{x\to 2} \dfrac{x^2-1}{x^2+1}$

k. $\lim\limits_{n\to 0} \dfrac{2^n}{4^n}$

l. $\lim\limits_{x\to 0} \dfrac{5x^2-2x+3}{7x^2+4x-3}$

m. $\lim\limits_{x\to 3} \dfrac{1}{\quad}$

n. $\lim\limits_{x\to 0} \dfrac{4-\sqrt{12+x}}{\quad}$

o. $\lim\limits_{x\to 0} \dfrac{\sqrt{x+5}-\sqrt{5-x}}{x}$

p. $\lim\limits_{x\to -3} \dfrac{x^2+3x}{x+3}$

q. $\lim\limits_{x\to -2} \dfrac{x^3+x^2-8x-12}{x+2}$

r. $\lim\limits_{x\to 2} \dfrac{x^3+x^2-12}{x-2}$

s. $\lim\limits_{x\to -4} \dfrac{64+x^3}{4+x}$

t. $\lim\limits_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{2+x}-\dfrac{1}{2}\right)$

u. $\lim\limits_{x\to -1} \dfrac{108(x^2+2x)(x+1)^3}{\quad}$

18. Explain why the given limit does not exist.

a. $\lim\limits_{x\to 3} \sqrt{x-3}$

b. $\lim\limits_{x\to 2} \dfrac{1}{\quad}$

c. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-4x+4}$

d. $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

e. $f(x) = \begin{cases} -5, & x < 1 \\ 2, & x \geq 1 \end{cases}$; $\lim\limits_{x\to 1} f(x)$

f. $f(x) = \begin{cases} 5x^2, & x < -1 \\ 2x+1, & x \geq -1 \end{cases}$; $\lim\limits_{x\to -1} f(x)$

19. Write an essay about Sir Isaac Newton and his discovery of calculus.

20. Write an essay about Gottfried Leibniz and his discovery of calculus.

21. Write an essay about the controversy surrounding the discovery of calculus by Newton and Leibniz.

Chapter 3 Test

Achievement Category	Questions
Knowledge/Understanding	5–9, 11–14
Thinking/Inquiry/Problem Solving	4, 17
Communication	1, 2, 3
Application	10, 15, 16

1. Explain how you find the limit of a polynomial function.

2. Explain how you find the limit of a rational function.

3. Explain why the $\lim_{x \to 1} \frac{1}{x - 1}$ does not exist.

4. Give an example for which neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists, but $\lim_{x \to a} [f(x) + g(x)]$ does exist.

5. Consider the graph of the function $f(x) = 5x^2 - 8x$. Find the slope of the secant that joins the points on the graph given by $x = -2$ and $x = 1$.

6. State the slope of the line perpendicular to $y = \frac{3}{4}x + 5$.

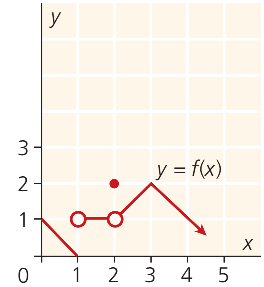
7. State the y-intercept of the function $f(x) = \frac{\sqrt{x^2 + 100}}{5}$.

8. State the equation of the line through $(0, -2)$ with a slope of -1.

9. For the function in the diagram, find the following:

   a. $\lim_{x \to 1} f(x)$ b. $\lim_{x \to 2} f(x)$ c. $\lim_{x \to 4^-} f(x)$

   d. values of x for which f is discontinuous.



10. The population of a city grows from 100 000 people to an amount P given by the formula $P = 100\,000 + 4000t$, where t is measured in years.

    a. Find the number of people in the city in 20 years.

    b. Determine the growth rate, in people per year, at $t = 10$ years.

11. A weather balloon is rising vertically. After t hours, its distance, measured in kilometres above the ground, is given by the formula $s(t) = 8t - t^2$.

    a. Determine its average velocity from $t = 2$ h to $t = 5$ h.

    b. Find its velocity at $t = 3$ h.

12. Find the average rate of change of $f(x) = \sqrt{x + 11}$ with respect to x from $x = 5$ to $x = 5 + h$.

13. Find the slope of the tangent at $x = 4$ for the function $f(x) = \frac{x}{x^2 - 15}$.

14. Find the following limits:

    a. $\lim\limits_{x\to3}\frac{4x^2 - 36}{2x - 6}$

    b. $\lim\limits_{x\to2}\frac{2x^2 - x - 6}{3x^2 - 7x + 2}$

    c. $\lim\limits_{x\to5}\frac{x - 5}{}$

    d. $\lim\limits_{x\to-1}\frac{x^3 + 1}{x^4 - 1}$

    e. $\lim\limits_{x\to3}\left(\frac{1}{x - 3} - \frac{6}{x^2 - 9}\right)$

    f. $\lim\limits_{x\to0}\frac{(x + 8)^{\frac{1}{3}} - 2}{x}$

15. Find constants a and b such that the function $f(x) = \begin{cases} ax + 3, \text{ if } x > 5 \\ 8, \text{ if } x = 5 \\ x^2 + bx + a, \text{ if } x < 5 \end{cases}$

    is continuous for all x.

16. Sketch a graph of a function with the following properties:

    a. $f(0) = 3$ b. $\lim\limits_{x\to1^+} f(x) = 3$ c. $\lim\limits_{x\to1^-} f(x) = 4$ d. $f(2) = -1$

17. For what value of k is the following a continuous function?

$$f(x) = \begin{cases} \dfrac{x - 2}{}, \text{ if } x \geq -\frac{2}{7} \text{ and } x \neq 2 \\ k, \text{ if } x = 2 \end{cases}$$